TME35 Mechanics of Solids, 14th September 2012

Assignments A1-A13

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Assignment 1

Rewriting the expression in *(a)* using index notation is fairly straight-forward, as it becomes $b_{ij}c_j d_i$. Expanding the expression in *(b)* is tedious:

$$\begin{aligned} a_{ijk}b_{ik} &= a_{1i1}b_{11} + a_{1i2}b_{12} + a_{1i3}b_{13} \\ &+ a_{2i1}b_{21} + a_{2i2}b_{22} + a_{2i3}b_{23} \\ &+ a_{3i1}b_{31} + a_{3i2}b_{32} + a_{3i3}b_{33} \end{aligned}$$

Assignment 2

Defining $\boldsymbol{F}_g = (0, 0, -mg)$ and $\boldsymbol{F}_A$ according to figure 1, we can assume that around the origin, we will have a moment equilibrium, i.e. $\boldsymbol{r}_A \times \boldsymbol{F}_A + \boldsymbol{r}_{A/2} \times \boldsymbol{F}_g = \boldsymbol{0}$. This gives

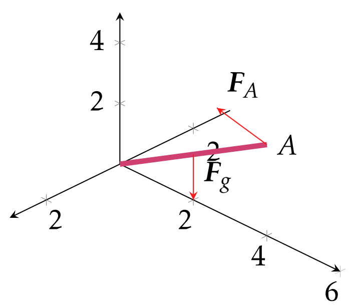

Figure 1: *The rod and forces affecing it.*

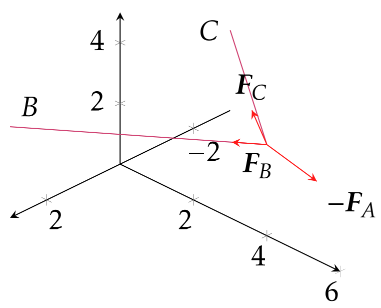

Figure 2: *The two cables with forces.*

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$$\boldsymbol{r}_A \times \boldsymbol{F}_A + \boldsymbol{r}_{A/2} \times \boldsymbol{F}_g = [a(4f_z - 3f_y), 3af_x, -4af_x] - [-2amg, 0, 0] = \\ = a[4f_z - 3f_y - 2mg, 3f_x, -4af_x] = \mathbf{0},$$

which in turn means that $f_x = 0$ and $4f_z - 3f_y - 2mg = 0$. Knowing that according to figure 2 on the previous page we must have $\boldsymbol{F}_B + \boldsymbol{F}_C - \boldsymbol{F}_A = 0$, and that both $\boldsymbol{F}_B$ and $\boldsymbol{F}_C$ must be parallel to the cables they affect, we can assume that $\boldsymbol{F}_B = b[3, -4, 0]$ and $\boldsymbol{F}_C = c[-1, -2, 1]$. This gives us a system of equations

$$\begin{cases} 3b - 1c = 0 \\ -4b - 2c - \frac{f_y}{a} = 0, \\ 1c - \frac{f_z}{a} = 0 \end{cases}$$

which is partially solved with $c = 3b$. Combining the remaining part of this system with the relation $4f_z - 3f_y - 2mg = 0$ we got earlier, and setting $f_z = zamg$, $f_y = yamg$, we get a new system

$$\begin{cases} -10b - ymg = 0 \\ 3b - zmg = 0, \\ 4z - 3y - 2 = 0 \end{cases}$$

which is completely solvable with $y = -\frac{3}{10}z = -\frac{10b}{mg}$, $b = \frac{20}{147}mg$ and $c = \frac{60}{147}mg$. Given this, we can calculate the forces as

$$\boldsymbol{F}_B = amg\left[\frac{60}{147}, -\frac{80}{147}, 0\right], \\ \boldsymbol{F}_C = amg\left[-\frac{60}{147}, -\frac{120}{147}, \frac{60}{147}\right].$$

Assignment 3

Transforming the coordinates is as easy as calculating $b'_i = \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j b_j = l_{ij} b_j$, where

$$[l_{ij}] = \begin{bmatrix} \cos\left(-\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{3}\right) & 0 \\ \cos\left(-\frac{2\pi}{3}\right) & \cos\left(\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

resulting in $\boldsymbol{b} = [2 - \sqrt{3}/2, 1/2 + 2\sqrt{3}, 3]^T$.

Assignment 4

Proving the given statement can be done by proving $A_{ik}A^{-1}_{kj} = \delta_{ij}$:

$$A_{ik}A^{-1}_{kj} = (B_{ik} + \alpha u_i v_k)\left(B^{-1}_{kj} - \frac{\alpha}{1 + \alpha v_l B^{-1}_{lm} u_m} B^{-1}_{kn} u_n v_o B^{-1}_{oj}\right) =$$
$$= B_{ik}B^{-1}_{kj} + B^{-1}_{ik}\alpha u_i v_k - \alpha \frac{B^{-1}_{ik} B_{kn} u_n v_o B^{-1}_{oj} + \alpha u_i v_k B^{-1}_{kn} u_n v_o B^{-1}_{oj}}{1 + \alpha v_l B^{-1}_{lm} u_m} =$$
$$= \delta_{ij} + \alpha B^{-1}_{ik} u_i v_k - \alpha \frac{\delta_{in} u_n v_o B^{-1}_{oj} + u_i v_o B^{-1}_{oj} \alpha v_k B^{-1}_{kn} u_n}{1 + \alpha v_l B^{-1}_{lm} u_m} =$$
$$= \delta_{ij} + \alpha B^{-1}_{ik} u_i v_k - \alpha u_i v_o B^{-1}_{oj} \frac{1 + \alpha v_k B^{-1}_{kn} u_n}{1 + \alpha v_l B^{-1}_{lm} u_m} = \delta_{ij}$$

Assignment 5

Showing *(a)* is fairly simple. Knowing that $(\lambda \hat{\boldsymbol{n}})^{(k)} = \boldsymbol{T} \cdot \hat{\boldsymbol{n}}^{(k)}$, we manipulate the expression as follows:

$$\begin{aligned}
(\lambda \hat{n}_i)^{(k)} = T_{ij} \hat{n}^{(k)}_j &\implies (\lambda \hat{n}_i)^{(k)} \hat{n}^{(l)}_j = T_{ij} \hat{n}^{(k)}_j \hat{n}^{(l)}_j \\
&\implies (\lambda \hat{n}_i)^{(k)} \hat{n}^{(l)}_j = T_{ij} \delta_{kl} \\
&\implies (\lambda \hat{n}_i)^{(k)} \hat{n}^{(k)}_j = T_{ij} \\
&\implies \boldsymbol{T} = (\lambda \hat{\boldsymbol{n}})^{(k)} \hat{\boldsymbol{n}}^{(k)}
\end{aligned}$$

Further, showing *(b)* is not that difficult either, assuming $\boldsymbol{T}^k = \boldsymbol{T} \cdot \ldots \cdot \boldsymbol{T}$. Since the vectors $\hat{\boldsymbol{n}}^{(i)}$ are orthonormal, all products resulting from $\boldsymbol{T}^k$ will be either 0 or $\left(\lambda^{(i)}\right)^k \hat{\boldsymbol{n}}^{(i)} \hat{\boldsymbol{n}}^{(i)}$:

$$\exp(\boldsymbol{T}) = \sum_{k=0}^{\infty} \frac{\boldsymbol{T}^k}{k!} = \sum_{k=0}^{\infty} \frac{\left((\lambda \hat{\boldsymbol{n}})^{(i)} \hat{\boldsymbol{n}}^{(i)}\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(\lambda^{(i)}\right)^k}{k!} \hat{\boldsymbol{n}}^{(i)} \hat{\boldsymbol{n}}^{(i)} = \exp\left(\lambda^{(i)}\right) \hat{\boldsymbol{n}}^{(i)} \hat{\boldsymbol{n}}^{(i)}.$$

Using this relation on the given matrix, which has eigenvalues 2 and $(9 \pm \sqrt{73})/2$, yields

$$\exp(\boldsymbol{T}) = \begin{bmatrix} \frac{1}{32} e^{\frac{9}{2} - \frac{\sqrt{73}}{2}} \left(41 - 3\sqrt{73} + \left(41 + 3\sqrt{73}\right) e^{\sqrt{73}}\right) & \frac{1}{8} e^{\frac{9}{2} - \frac{\sqrt{73}}{2}} \left(3 - \sqrt{73} + \left(3 + \sqrt{73}\right) e^{\sqrt{73}}\right) & 0 \\ \frac{1}{8} e^{\frac{9}{2} - \frac{\sqrt{73}}{2}} \left(3 - \sqrt{73} + \left(3 + \sqrt{73}\right) e^{\sqrt{73}}\right) & e^{\frac{9}{2} - \frac{\sqrt{73}}{2}} \left(1 + e^{\sqrt{73}}\right) & 0 \\ 0 & 0 & e^2 3 \end{bmatrix}$$

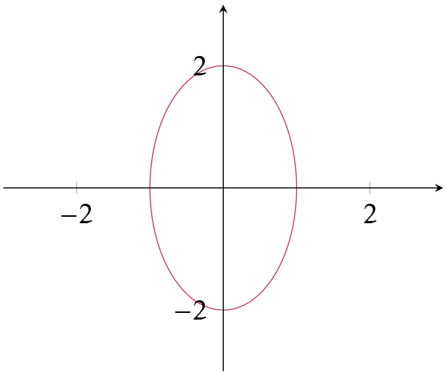

Figure 3: *Illustrating the equation $\Phi = 0$*

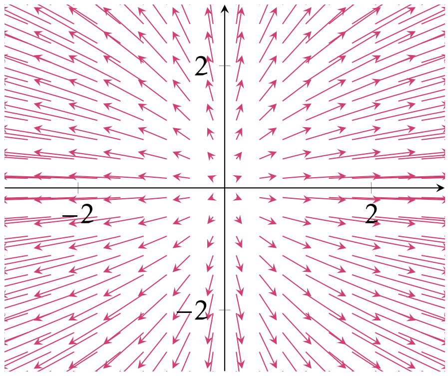

Figure 4: *Illustrating the vector field $\nabla\Phi$.*

Assignment 6

The equation $\Phi = x_1^2 + \left(\frac{x_2}{2}\right)^2 - 1 = 0$ can be visualized as an ellipse centered at the origin, with foci at $[0, \pm\sqrt{3}]$ (as in figure 3). The vector field $\nabla\Phi$ is illustrated by the quiver plot in figure 4.

Assignment 7

Proving the formula is easy using Gauss' divergence theorem and applying the chain rule of differentiation:

$$\oint_\Gamma \hat{n}_j \sigma_{ij} \varphi_i \,\mathrm{d}s = \int_\Omega \partial_j(\sigma_{ij}\varphi_i)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \sigma_{ij}\partial_i\varphi_i \,\mathrm{d}\boldsymbol{x} + \int_\Omega \varphi_i \partial_j \sigma_{ij}\,\mathrm{d}\boldsymbol{x}$$

$$\implies \int_\Omega \varphi_i \partial_j \sigma_{ij}\,\mathrm{d}\boldsymbol{x} = \oint_\Gamma \hat{n}_j \sigma_{ij}\varphi_i\,\mathrm{d}s - \int_\Omega \sigma_{ij}\partial_i\varphi_i\,\mathrm{d}\boldsymbol{x}$$

Assignment 8

Using the relation $\sigma'_{ij} = \sigma_{ij} = \sigma_m \delta_{ij}$ and knowing that in our case $\sigma_m = 30$, we can easily obtain

$$[\boldsymbol{\sigma}'] = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & 10 \\ 10 & 10 & 0 \end{bmatrix},$$

with principal values (stresses) 0 and $\pm 10\sqrt{2}$ with corresponding principal vectors (directions) $[-1, 1, 0]$ and $[\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 1]$.

The invariants can be calculated as $I_1 = I_3 = 0$ and $I_2 = -200$, and the stress vector with respect to the given plane would be $\boldsymbol{t} = \hat{\boldsymbol{n}}\boldsymbol{\sigma} = [0, 0, 20 + 2\sqrt{5}]$.

Assignment 9

With $V = \int_\Omega \mathrm{d}\boldsymbol{x}$, we have

$$\lim_{V\to 0} \frac{\dot{V}}{V} = \lim_{V\to 0} \frac{1}{V} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathrm{d}\boldsymbol{x} = \lim_{V\to 0} \frac{1}{V} \int_\Omega \left(\frac{\mathrm{d}}{\mathrm{d}t} 1 + 1 v_{i,i} \right) \mathrm{d}\boldsymbol{x} =$$

$$= \lim_{V\to 0} \frac{1}{V} \int_\Omega v_{i,i} \,\mathrm{d}\boldsymbol{x} = \lim_{V\to 0} \frac{1}{V} v_{i,i} V = v_{i,i},$$

where the second to last step takes advantage of the upper and lower bounds of the integral converging towards $v_{i,i}$ as V approaches 0 (and Ω approaches a single point).

Assignment 10

The principle of angular momentum is given in tensor form as

$$\oint_\Gamma e_{ijk} x_i t_j \,\mathrm{d}s + \int_\Omega \rho e_{ijk} x_i f_j \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho e_{ijk} x_i v_j \,\mathrm{d}V.$$

We apply the relation $t_j = n_p \sigma_{pj}$ as well as the divergence theorem and Reynold's theorem to obtain

$$\int_\Omega e_{ijk} \left(x_i \sigma_{pj} \right)_{,p} \mathrm{d}V + \int_\Omega \rho e_{ijk} x_i f_j \,\mathrm{d}V = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} \left(\rho e_{ijk} x_i v_j \right) + \left(\rho e_{ijk} x_i v_j \right) v_{l,l} \,\mathrm{d}V.$$

We continue consolidating these integrals (also using mass conservation to remove some terms from the right-hand side):

$$\int_\Omega e_{ijk} \left(x_i \sigma_{pj,p} + \delta_{ip} \sigma_{pj} + \rho x_i f_j \right) \mathrm{d}V = \int_\Omega e_{ijk} \left(\dot{\rho} x_i v_j + \rho x_i \dot{v}_i + \rho v_{l,l} x_i v_j \right) \mathrm{d}V$$

$$\Longrightarrow \int_\Omega e_{ijk} \left(x_i \left(\sigma_{pj,p} + \rho f_j - \dot{v}_j \right) + \sigma_{ij} \right) \mathrm{d}V = 0$$

This implies that $e_{ijk}\sigma_{ij} = 0$, which in turn implies that $\boldsymbol{\sigma}$ is a symmetric tensor.

Assignment 11

Using $\dot{K} + \dot{U} = \frac{\mathrm{d}W}{\mathrm{d}t} + \frac{\mathrm{d}H}{\mathrm{d}t}$ along with the tensor representations of these,

$$\begin{aligned}
\dot{K} &= \int_\Omega v_i \left(\sigma_{ji,j} + \rho f_i\right) \mathrm{d}V, \\
\dot{U} &= \int_\Omega \rho \dot{e} \, \mathrm{d}V, \\
\frac{\mathrm{d}W}{\mathrm{d}t} &= \int_\Omega \left(\rho f_i v_i + \left(v_i \sigma_{ji}\right)_{,j}\right) \mathrm{d}V, \\
\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_\Omega \left(\rho \mathcal{E} - q_{i,i}\right) \mathrm{d}V,
\end{aligned}$$

we can substitute and consolidate the integrals to arrive at *(121)*:

$$\begin{aligned}
&\int_\Omega v_i(\sigma_{ji,i} + \rho f_i) + \rho \dot{e} - (\rho \mathcal{E} - q_{i,i}) - (\rho f_i v_i + (v_i \sigma_{ji})_{,j}) \, \mathrm{d}V = 0 \\
&\implies \int_\Omega \rho \dot{e} - (v_{i,j}\sigma_{ji} + v_i \sigma_{ji,j} - v_i \sigma_{ji,j}) + q_{i,i} - \rho \mathcal{E} \, \mathrm{d}V = 0 \\
&\implies \int_\Omega \rho \dot{e} - v_{i,j}\sigma_{ij} + q_{i,i} - \rho \mathcal{E} \, \mathrm{d}V = 0,
\end{aligned}$$

which is what we wanted to show.

Assignment 12

Given $p_{\text{mech}} = p(\rho, \theta)$, we must have $p(\rho,\theta) = -\frac{1}{3}\sigma_{ii} + (\lambda^* + \frac{2}{3}\mu^*)D_{kk}$. This gives us

$$\begin{aligned}
\sigma_{ij} &= -p(\rho,\theta)\delta_{ij} + \tau_{ij} \\
\sigma_{ij} &= -p(\rho,\theta)\delta_{ij} + \lambda^*\delta_{ij}D_{kk} + 2\mu^* D_{ij} \\
\sigma_{ij} - \frac{1}{3}\sigma_{ii}\delta_{ij} &= 2\mu^* D_{ij} - \frac{2}{3}\mu^* D_{kk} \\
\sigma'_{ij} &= 2\mu^* D'_{ij}.
\end{aligned}$$

Assignment 13

Combining Hooke's law, $\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i})$, and the momentum equation $\sigma_{ij,j} + \rho f_i = \rho\dot{v}_i$, we get

$$\begin{aligned}
\sigma_{ij,j} + \rho f_i &= \rho\dot{v}_i \\
(\lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}))_{,j} + \rho f_i &= \rho\ddot{u}_i \\
\lambda\delta_{ij}u_{k,kj} + \mu(u_{i,jj} + u_{j,ij}) + \rho f_i &= \rho\ddot{u}_i \\
\lambda u_{j,ji} + \mu(u_{i,jj} + u_{j,ji}) + \rho f_i &= \rho\ddot{u}_i \\
\mu u_{i,jj} + (\lambda + \mu)u_{j,ji}\rho f_i &= \rho\ddot{u}_i,
\end{aligned}$$

which is what we wanted to show.