TMA375 Partial differential equations
Assignment 2

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1 The heat equation

The heat equation describes the distribution of heat in an object over time, given a starting distribution $u(x,0) = f(x)$, but can also be used to describe and solve other fenomena and equations such as the famous *Black-Scholes* equation. A generalized variant can be used to describe diffusion in chemical processes.

The heat equation in two (and three) dimensions is given by (1.1), where $\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ as usual, and $\partial\Omega$ is the boundary of Ω. It is parabolic, and can be solved using Fourier series.

$$\begin{aligned} u_t - \Delta u &= 0, \quad x \in \Omega, \qquad T > 0 \\ u &= 0, \quad x \in \partial\Omega, \qquad T > 0 \\ u(x,0) &= u_0, \quad x \in \Omega \end{aligned} \tag{1.1}$$

1.1 Stability estimates

We can show two stability estimates for the heat equation:

$$||u(x,T)||^2 + \int_0^T ||\nabla u||^2 \, \mathrm{d}t \leq ||u_0||^2, \quad T > 0 \tag{1.2a}$$

$$||\Delta u(x,T)|| \leq \frac{1}{T} ||u_0||, \quad T > 0 \tag{1.2b}$$

To show (1.2a), we take the inner product of (1.1) and u, giving us

$$0 = \langle u_t, u\rangle + \langle -\Delta u, u\rangle = \frac{\mathrm{d}\,||u||^2}{\mathrm{d}t} + \langle -\Delta u, u\rangle \geq \frac{\mathrm{d}\,||u||^2}{\mathrm{d}t} + 2\,||\nabla u||^2$$

To get rid of the horrible-looking differentiation with respect to t, we simply integrate with respect to that variable, over the whole time interval $0 \leq t \leq T$:

$$\begin{aligned} &\int_0^T \frac{\mathrm{d}\,||u||^2}{\mathrm{d}t} \,\mathrm{d}t + 2\int_0^T ||\nabla u||^2 \,\mathrm{d}t \leq 0 \\ \implies &\left[||u||^2\right]_0^T + 2\int_0^T ||\nabla u||^2 \,\mathrm{d}t \leq 0 \\ \implies &||u(x,T)||^2 - ||u(x,0)||^2 + 2\int_0^T ||\nabla u||^2 \,\mathrm{d}t \leq 0 \\ \implies &||u(x,T)||^2 + 2\int_0^T ||\nabla u||^2 \,\mathrm{d}t \leq ||u_0||^2 \\ \implies &||u(x,T)||^2 + \int_0^T ||\nabla u||^2 \,\mathrm{d}t \leq ||u_0||^2 \end{aligned}$$

Thus, we have shown (1.2a). To show the second estimate (1.2b), we take the inner product with $t^2\Delta^2 u$ instead:

$$\langle t^2\Delta^2 u, u_t\rangle - \langle t^2\Delta^2 u, \Delta u\rangle = 0$$
$$\implies \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2\,||\Delta u||^2\right) - t^2\left(\Delta^2 u,\Delta u\right) = t\,||\Delta u||^2$$

Note that t isn't affected by the operator Δ, and therefore can be extracted from inner products as a constant. From this, (1.2b) easily follows by integration:

$$\frac{1}{2}\left[t^2\,||\Delta u||^2\right]_0^T + \int_0^T t^2\left(\Delta^2 u,\Delta u\right)\,\mathrm{d}t = \int_0^T t\,||\Delta u||^2\,\mathrm{d}t \leq \frac{1}{4}\,||u_0||^2$$
$$\implies \frac{T^2}{2}\,||\Delta u(x,T)||^2 + \int_0^T t^2\left(\Delta^2 u,\Delta u\right)\,\mathrm{d}t \leq \frac{1}{4}\,||u_0||^2$$
$$\implies \frac{T^2}{2}\,||\Delta u(x,T)||^2 \leq \frac{1}{4}\,||u_0||^2$$
$$\implies ||\Delta u(x,T)||^2 \leq \frac{1}{2T^2}\,||u_0||^2$$
$$\implies ||\Delta u(x,T)|| \leq \frac{1}{\sqrt{2}T}\,||u_0|| \leq \frac{1}{T}\,||u_0||$$

This proves (1.2b). In the first step, the inequality $\int_0^T t\,||\Delta u)||^2\,\mathrm{d}t \leq \frac{1}{4}\,||u_0||^2$ is used. This inequality can be obtained by taking the inner product of $u_t - \Delta u = 0$ and $-t\Delta u$, and using the symmetry of the Δ operator.

1.2 Substituting Δu with $u_{xx} + 4u_{yy}$

Other interesting partial differential equations sometimes arise that are similar to the heat equation. It is interesting to see what happens to stability estimates when the operator Δ is replaced by something else, for instance $u_{xx} + 4u_{yy}$, as in (1.3).

$$\begin{aligned} u_t - (u_{xx} + 4u_{yy}) = 0, &\quad x \in \Omega, \qquad T > 0 \\ u = 0, &\quad x \in \partial\Omega, \qquad T > 0 \\ u(x,0) = u_0, &\quad x \in \Omega \end{aligned} \tag{1.3}$$

Clearly, the estimates won't be the same. Now, let us define the operator $A = \frac{\partial^2}{\partial x^2} + 4\frac{\partial^2}{\partial y^2}$ and use that in our calculations instead. Then, according to Eriksson et. al. (1996, p. 405), our stability estimates are as follows:

$$||u(x,T)||^2 + 2\int_0^T \langle Au,u\rangle\,\mathrm{d}t = ||u_0||^2 \tag{1.4a}$$
$$||Au(x,T)|| \leq \frac{1}{\sqrt{2}T}\,||u_0|| \tag{1.4b}$$

With our A, the first estimate becomes slightly more cumbersome, replacing a norm with an actual inner product. This cannot be simplified easily, since there is no operator equivalent to "$\sqrt{A}$" as with Δ. The second estimate remains pretty much the same.

By scaling the y coordinate appropriately (by a fourth in our case) we can reduce the problem to the heat equation given by (1.1) and use the original stability estimates. This imples that any problem that is similar to the heat equation can be solved by it, and that they are as stable. This is an important result; such equations are called *parabolic* PDEs.

1.3 A Fourier series solution with $\Omega = [0,1]$

In one dimension, with $\Omega = [0,1]$, (1.1) is given by:

$$\begin{aligned} u_t - u_{xx} &= 0, \quad 0 \le x \le 1, \qquad T > 0 \\ u(0,t) = u(1,t) &= 0, \qquad\qquad\qquad\quad T > 0 \\ u(x,0) &= u_0, \quad 0 \le x \le 1 \end{aligned} \tag{1.5}$$

This partial differential equation can be solved by separation of variables and Fourier series. One simply conjectures that the solution is of the form $u(x,t) = X(x)T(x)$ where $X(0) = X(1) = 0$. Substituting into 1.5 and separating dependance on x and t gives us an eigenvalue problem (1.6) and an initial value problem (1.7) instead.

$$\begin{cases} -X''(x) = \lambda X(x) & \text{for } t > 0, \\ X(0) = X(1) = 0 \end{cases} \tag{1.6}$$

$$\begin{cases} T'(t) = -\lambda T(t) & \text{for } t > 0, \\ T(0) = 1 \end{cases} \tag{1.7}$$

The eigenvalue problem (1.6) is solved by eigenfunctions $X_i(x) = \sin(\pi k x)$ with corresponding eigenvalues $\lambda_k = k^2$, $k = 1,2,\ldots$ — for each of these eigenvalues, the initial value problem given by (1.7) can be solved, giving us a solution for $T(t)$ given by $T(t) = e^{-j^2 t}$.

This gives us a set of solutions $\{X_k(x)T(t);\ k = 1,2,\ldots\}$ which we can use to provide a solution to our specific problem, assuming $u_0(x)$ has a convergent fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} a_k X_k(x)T(t) = \sum_{k=1}^{\infty} a_k e^{-j^2 t} \sin(\pi k x) \tag{1.8}$$

The coefficients a_k are given by the Fourier series of $u_0(x)$ in the domain in question, i.e. by the explicit formula $a_k = 2\int_0^1 u_0(x)\sin(kx)\,\mathrm{d}x$.

The smoothing property of the heat equation can be directly related to the set if solutions given above. It's obvious that $X(x)$ is perioﬁc and bounded, and that it only takes values on the interval $[-1,1]$. However, $T(t)$ is not periodic. It is, however, bounded and only takes on values on the interval $[0,1]$ — it is also monotonically decreasing (and quickly so), with $\lim_{t\to\infty} T(t) = 0$. Thus our solution, given by (1.8), will decrease monotonically and very quickly with respect to t, and will eventually reach a state where $u(x,T) = 0, \forall x$.

2 Application of PDE: fluid flow

An equation similar to the heat equation is the equation describing incompressible and irrotational (turbulence-free) flow of fluids:

$$\begin{aligned} u = \nabla\phi, \quad \nabla u = 0 \\ \implies \nabla u = \nabla \cdot (\nabla\phi) = \Delta\phi = 0 \end{aligned} \tag{2.9}$$

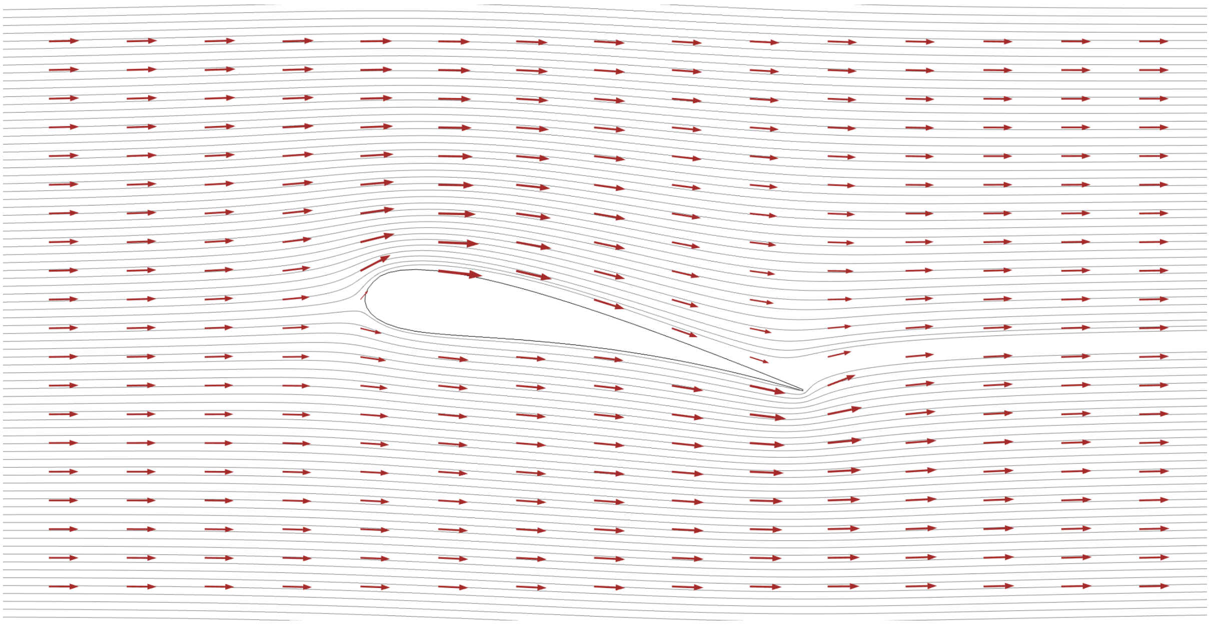

Figure 1: The flow u over an airfoil (wing) with an angle of attack $\alpha = 10°$

This is very similar to the heat equation — in fact, for certain boundary conditions, (2.9) is equivalent to (1.1). With suitable boundaries and boundary conditions, this can be made interesting; one could simulate fluid flow around an airfoil as seen in figure 1. This is easily done in COMSOL — the domain is specified by and excessively large area with a hole the shape of an airfoil in the middle. One then sets suitable boundary conditions, i.e. Neumann conditions on the boundaries around the airfoil, and Dirichlet conditions on those boundaries through which the fluid enters and exits. What we're simulating is therefore essentially an airfoil inside a large tube through which incompressible and irrotational fluid flows.

2.1 Model shortcomings

This model has several shortcomings, however. Firstly, almost no fluids (or gases) are incompressible. None of them are also irrotational. Thus, the model doesn't represent reality and cannot be used to simluate real-world phenomena reliably.

It also forces the flow into a "tube"; an unrealistic situation for general use. This is due to the fact that one cannot set up a FEM formulation for infinite domains. One could avoid this problem by simply omitting conditions on boundaries.

As seen in figure 1, the flow moves *upwards* from the airfoil when leaving the back of it. This is contrary to how airfoils actually work; "pushing" air downwards to benefit from the fact that every force has an equal and opposite reaction — thus making the airfoil "lift" into the air. This is likely due to the fact that the model describes fluid as incompressible and irrotational, meaning neither pressure or turbulence are taken into account.

References

Eriksson, K., Estep, D., Hansbo, P., Johnson, C. (1996) *Computational Differential Equations.* Lund: Studentlitteratur.